

ON A THEORY OF THE b -FUNCTION IN POSITIVE CHARACTERISTIC

THOMAS BITOUN

ABSTRACT. We introduce a theory of the b -function (or Bernstein-Sato polynomial) in positive characteristic. Let k be a field of characteristic $p > 0$ and let $f \in k[x_1, \dots, x_n]$ be a nonconstant polynomial. The b -function of f is an ideal of a nonnoetherian commutative algebra of characteristic p , but it has “roots” in \mathbb{Z}_p . We prove the existence of the b -function as well as the rationality of its roots. The framework of the theory is that of unit F -modules. There is a close connection with test ideals. In particular, we prove that the roots of the b -function are the opposites of the F -jumping exponents of f which are in $(0, 1] \cap \mathbb{Z}_{(p)}$.

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INTRODUCTION

Let $Q \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial. The b -function or Bernstein-Sato polynomial of Q ([2], [14]) is of foundational importance in \mathcal{D} -module theory. It is related to ζ -functions ([2], [15]), to the nearby and vanishing cycles of Q ([1], [7], [11]), as well as to many invariants of its singularities, see [13] for a survey.

The goal of this note is to present a theory of the b -function in positive characteristic.

Let us first recall the classical complex theory, see for example [9, Chapter 6].

The classical theory. Let $Q \in \mathbb{C}[x_1, \dots, x_n]$ be a nonconstant polynomial. We denote by \mathcal{D} the ring $\mathcal{D}_{\mathbb{A}^n}$ of polynomial differential operators on \mathbb{A}^n and let $\mathcal{D}[s] := \mathbb{C}[s] \otimes_{\mathbb{C}} \mathcal{D}$, where s is a central parameter.

For $U := \{Q \neq 0\} \subset \mathbb{A}^n$, set $\mathcal{D}_U[s]Q^s$ to be the following (left) $\mathcal{D}_U[s]$ -module. It is the free $\mathbb{C}[x_1, \dots, x_n][\frac{1}{Q}][s]$ -module with generator Q^s such that for a vector field v on U ,

$$v.Q^s := sv(Q)\frac{1}{Q}Q^s.$$

Definition. The $\mathcal{D}[s]$ -module $\mathcal{D}[s]Q^s$ is the $\mathcal{D}[s]$ -submodule of $\mathcal{D}_U[s]Q^s$ generated by Q^s .

According to J. Bernstein and M. Sato, the b -function or Bernstein-Sato polynomial of Q is the following complex polynomial in one variable.

Definition. The b -function of Q is the monic generator $b_Q(s)$ of the annihilator of the natural action of $\mathbb{C}[s]$ on the $\mathcal{D}[s]$ -module $\mathcal{D}[s]Q^s/\mathcal{D}[s]QQ^s$.

Below, we will write $\mathcal{D}[s]Q^{s+1}$ instead of $\mathcal{D}[s]QQ^s$.

Remark. The existence of a nonzero polynomial $\in \mathbb{C}[s]$ annihilating $\mathcal{D}[s]Q^s/\mathcal{D}[s]Q^{s+1}$ is not trivial. It is a consequence of the holonomicity of the \mathcal{D} -module $\mathcal{D}[s]Q^s/\mathcal{D}[s]Q^{s+1}$, which is the basic result of the theory, see e.g. [9, Theorem 6.7.].

The b -function satisfies the following rationality and negativity theorem, due to Kashiwara ([8]):

Theorem. The roots of $b_Q(s)$ are negative rational numbers.

These are the main theorems of the classical theory.

In order to present our results, we would like to rephrase the definition of the b -function, after [11].

For a commutative ring A and an element $a \in A$, we denote by A_a the localization of A obtained by inverting a . Let $A = \mathbb{C}[x_1, \dots, x_n]$ in what follows.

Let $B_Q := A[t]_{Q-t}/A[t]$. It naturally is a left $\mathcal{D}_{\mathbb{A}^n \times \mathbb{A}^1}$ -module, where t is a coordinate of the extra \mathbb{A}^1 . We then have:

Proposition. There is an injective σ -linear morphism $\gamma :$

$$\begin{aligned} \mathcal{D}_U[s]Q^s &\rightarrow A_Q \otimes_A B_Q \\ Q^s &\mapsto \left[\frac{1}{Q-t}\right] \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_U[s] &\xrightarrow{\sigma} \mathcal{D}_{A_Q[t]} \\ P(s) &\mapsto P(-\partial_t t). \end{aligned}$$

Moreover γ induces an isomorphism

$$\mathcal{D}[s]Q^s \cong M_Q := \mathcal{D}[\partial_t t] \cdot \left[\frac{1}{Q-t}\right] \subset B_Q$$

which maps $\mathcal{D}[s]Q^{s+1}$ isomorphically to tM_Q .

Thus γ induces a \mathcal{D} -module isomorphism:

$$\mathcal{D}[s]Q^s/\mathcal{D}[s]Q^{s+1} \rightarrow M_Q/tM_Q,$$

under which the action of s transports to that of $-\partial_t t$. Hence we can reformulate the definition of the b -function as follows:

Definition. *The b -function of Q is the monic generator b_Q of the annihilator of the natural action of $\mathbb{C}[-\partial_t t]$ on the $\mathcal{D}[\partial_t t]$ -module M_Q/tM_Q .*

Our definition of the b -function in positive characteristic is analogous.

Positive characteristic. Let k be a field of characteristic $p > 0$. Let $f \in k[x_1, \dots, x_n] =: R$ be a nonconstant polynomial. Then $B_f := R[t]_{f-t}/R[t]$ is naturally a left $D_{R[t]}$ -module, where $D_{R[t]}$ is the Grothendieck ring of differential operators on $\mathbb{A}^n \times \mathbb{A}^1$. Recall that $\forall e \geq 0$, $D_{R[t]}$ contains the differential operator $\partial_t^{[p^e]}$, whose action on t^l is $\partial_t^{[p^e]} t^l = \binom{l}{p^e} t^{l-p^e}$. Let $M_f := D_R[-\partial_t^{[p^e]} t^{p^e} | e \geq 0] \cdot [\frac{1}{f-t}] \subset B_f$.

We define

Definition. *(Definition 3) The b -function of f is the ideal b_f of $k[-\partial_t^{[p^e]} t^{p^e} | e \geq 0]$ annihilating M_f/tM_f , for the natural action of $k[-\partial_t^{[p^e]} t^{p^e} | e \geq 0]$ induced by that of $D_{R[t]}$ on B_f .*

We note the following:

Lemma. *(Remark 1) The maximal ideals of $k[-\partial_t^{[p^e]} t^{p^e} | e \geq 0]$ are in canonical bijection with the p -adic integers \mathbb{Z}_p .*

It is thus natural to make the subsequent definition.

Definition. *Let I be an ideal of $k[-\partial_t^{[p^e]} t^{p^e} | e \geq 0]$. The roots of I are the p -adic integers corresponding to the maximal ideals of $k[-\partial_t^{[p^e]} t^{p^e} | e \geq 0]$ containing I , by the lemma.*

We give two proofs of the existence of the b -function in positive characteristic. That is that the b -function has finitely many roots, see Corollaries 1 and 2. We also prove the following analogue of Kashiwara's Theorem:

Theorem. *(Corollary 2) The roots of the b -function of f are negative rational numbers, ≥ -1 .*

Remark. *Note that the positive characteristic theory differs from its complex counterpart in that the roots of the b -function are always ≥ -1 .*

The framework of our construction is the theory of unit F -modules ([10]) and the proofs are based on a relation to test ideals ([6]). In fact, our main result is a positive characteristic parallel to [4, Theorem B]:

Theorem. *(Theorem 5)*

The roots of the b -function of f are the opposites of the F -jumping exponents of f which are in $(0, 1] \cap \mathbb{Z}_{(p)}$.

Contents. In the first section, after considering the algebra of higher Euler operators, which we call the algebra of binomial coefficients, we give the definition of the b -function and show that the D -module M_f/tM_f underlies a finitely generated unit F -module. This is achieved by approximating by the level of differential operators and using the relation of these approximations to test ideals. Key there is the discreteness of the set of F -jumping exponents.

In the second section, we carry out a finer study of the relation to test ideals and prove the announced theorems. They ultimately rely on elementary properties of p -adic and $\frac{1}{p}$ -adic expansions of rational numbers, which we gather in a preliminary subsection.

We give some examples in the third section.

Acknowledgements. This theory of the b -function in characteristic p sprang from my attempt to understand the work of Mircea Mustața [12]. I thank him for suggesting the problem as well as answering many questions about test ideals. I would also like to thank Roman Bezrukavnikov and Pavel Etingof for interesting discussions.

1. THE CONSTRUCTION

1.1. The algebra of binomial coefficients. Let k be a field of characteristic $p > 0$, and let $D_{\mathbb{A}_k^1}$ be the full ring of differential operators on the affine line over k , with coordinate t . After [12], we set the *higher Euler operators* to be the global differential operators $\nu_e \in D_{\mathbb{A}_k^1}$, $\forall e \in \mathbb{N}$:

$$\nu_e := \frac{d^{[p^e]}}{dt} t^{p^e}.$$

Definition 1. Let k be a field of characteristic p . Then the algebra of binomial coefficients $k[(\binom{s}{p^0}), (\binom{s}{p^1}), (\binom{s}{p^2}), \dots]$ is the sub- k -algebra of $D_{\mathbb{A}_k^1}$ generated by the higher Euler operators ν_e , $e \in \mathbb{N}$. In this algebra, $\forall e \in \mathbb{N}$, we denote $-\nu_e$ by the binomial coefficient symbol $(\binom{s}{p^e})$.

Thus $k[(\binom{s}{p^0}), (\binom{s}{p^1}), (\binom{s}{p^2}), \dots]$ is the quotient of the polynomial ring in infinitely many variables $k[x_e; e \in \mathbb{N}]$ by the relations, $\forall e \in \mathbb{N}$, $x_e^p = x_e$.

The following remark is fundamental, and justifies the notation:

Remark 1. The maximal ideals of $k[(\binom{s}{p^0}), (\binom{s}{p^1}), (\binom{s}{p^2}), \dots]$ are canonically identified to the p -adic integers.

Proof. Let $\pi : \mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$ be the quotient map and let $s : \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{Z}$ be its usual section with image $\{0, 1, \dots, p-1\}$.

To a p -adic integer α , we associate a maximal ideal \mathfrak{m}_α of $k[(\binom{s}{p^0}), (\binom{s}{p^1}), (\binom{s}{p^2}), \dots]$. Let $\alpha = \sum_{e \in \mathbb{N}} \alpha_e p^e$, $\forall e \in \mathbb{N}$, $\alpha_e \in \{0, 1, \dots, p-1\}$, be the p -adic expansion. Define

a surjective \mathbb{F}_p -algebra morphism,

$$m_\alpha : \mathbb{F}_p\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right] \rightarrow \mathbb{F}_p$$

by $m_\alpha\left(\binom{s}{p^e}\right) := \pi(\alpha_e)$. We set $\mathfrak{m}_\alpha := \ker m_\alpha$.

To a maximal ideal \mathfrak{m} of $k\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right]$, we associate a p -adic integer $\alpha_\mathfrak{m}$. Note first that if K is an overfield of k , a k -algebra morphism

$$m : k\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right] \rightarrow K$$

is uniquely defined over \mathbb{F}_p . Namely if $i : k \rightarrow K$ is the canonical inclusion, there is a unique \mathbb{F}_p -algebra morphism

$$\tilde{m} : \mathbb{F}_p\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right] \rightarrow \mathbb{F}_p$$

such that $m = i \circ (k \otimes_{\mathbb{F}_p} \tilde{m})$. In particular, the image of m is k . Indeed, by the relations above, $\forall e \in \mathbb{N}$, the image $m\left(\binom{s}{p^e}\right)$ satisfies $m\left(\binom{s}{p^e}\right)^p = m\left(\binom{s}{p^e}\right)$ and hence is in \mathbb{F}_p . Let

$$m : k\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right] \rightarrow K := k\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right]/\mathfrak{m}$$

be the quotient map. Then $K = k$ and we set $\alpha_\mathfrak{m} := \sum_{e \in \mathbb{N}} s(\tilde{m}\left(\binom{s}{p^e}\right))p^e$.

These maps are inverse to each other.

□

Definition 2. Let I be an ideal in $k\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right]$. The roots of I are the p -adic integers corresponding by Remark 1 to the maximal ideals containing I .

1.2. Definition of the b -function. Let k be a field of positive characteristic $p > 0$, X a smooth variety over k and $f : X \rightarrow \mathbb{A}^1$ a nonconstant function on X .

The b -function b_f of f is an ideal in $k\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right]$. In order to define it, we would like to introduce some notations.

Suppose that X is affine and let $\mathcal{O}(X) = R$. Then

$$B_f := R[t]_{f-t}/R[t]$$

is naturally a left $D_{X \times \mathbb{A}^1}$ -module. Let δ be the class of $\frac{1}{f-t}$ and

$$M_f := D_X[\nu_e; e \in \mathbb{N}]\delta.$$

We obtain the corresponding notions for general X by gluing. One sees that tM_f is a left sub- $D_X[\nu_e; e \in \mathbb{N}]$ -module of M_f ([12, lemma 6.4]) and thus that the quotient

$$N_f := M_f/tM_f$$

is a left $D_X[\nu_e; e \in \mathbb{N}]$ -module and in particular a $k[\nu_e; e \in \mathbb{N}]$ -module i.e. a $k\left[\binom{s}{p^0}, \binom{s}{p^1}, \binom{s}{p^2}, \dots\right]$ -module. We then set:

Definition 3. Let k be a field of positive characteristic p , X a smooth variety over k and $f : X \rightarrow \mathbb{A}^1$ a nonconstant function on X . The b -function b_f of f is the annihilator of N_f in $k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$.

1.3. On the Frobenius structure of N_f . Let us show that the D_X -module N_f is a unit F -module. In order to do so, we consider the ring of differential operators as the inductive limit of the rings of differential operators of bounded level, and similarly for the modules of the theory.

Let l be a non negative integer and let $D_X^{(l)} \subset D_X$ be the sheaf of rings of differential operators with divided powers of level at most $p^{l+1} - 1$. We set

$$M_f^{(l)} := D_X^{(l)}[\nu_e; e \leq l]\delta \subset M_f.$$

One has that $tM_f^{(l)} \subset M_f^{(l)}$ ([12, lemma 6.4]) and we denote the quotient by

$$N_f^{(l)} := M_f^{(l)} / tM_f^{(l)}.$$

It is a left $D_X^{(l)}[\nu_e; e \leq l]$ -module. Note that, $\forall l \in \mathbb{N}$, we have natural inclusions $M_f^{(l)} \subset M_f^{(l+1)}$, inducing morphisms

$$N_f^{(l)} \rightarrow N_f^{(l+1)}.$$

Note that $M_f = \bigcup_{l \geq 0} M_f^{(l)}$ and $N_f = \varinjlim_{l \geq 0} N_f^{(l)}$, where the limit is taken over the above morphisms.

Let s be a global section of \mathcal{O}_X . We denote by $D_X^{(l)}s$ the left $D_X^{(l)}$ -submodule of \mathcal{O}_X generated by s .

By explicit computations, Mustașă observed the following:

Proposition 1. Let $l \in \mathbb{N}$. Then there is a natural isomorphism of left $D_X^{(l)}$ -modules,

$$M_f^{(l)} \cong \bigoplus_{0 \leq n < p^{l+1}} D_X^{(l)} f^n.$$

It induces an isomorphism:

$$N_f^{(l)} \cong \bigoplus_{0 \leq n < p^{l+1}} D_X^{(l)} f^n / D_X^{(l)} f^{n+1}.$$

Hence the higher Euler operators act on the right-hand side by transport of structure. The actions are as follow: If the base p expansion of n is $n = \sum_{0 \leq e < l+1} a_e p^e$, then $\forall e < l+1$, ν_e acts on $D_X^{(l)} f^n / D_X^{(l)} f^{n+1}$ by $-a_e$.

Finally, the natural inclusions $M_f^{(l)} \subset M_f^{(l+1)}$ transport to the injections:

$$\bigoplus_{0 \leq n < p^{l+1}} D_X^{(l)} f^n \rightarrow \bigoplus_{0 \leq m < p^{l+2}} D_X^{(l+1)} f^m$$

$$g \mapsto g \cdot \sum_{0 \leq j < p} (-1)^j \binom{p-1}{j} f^{jp^{l+1}}.$$

Proof. The first part of the statement is [12, Proposition 6.1.], the second is [12, Corollary 6.5.] and the last is [12, Remark 5.7.]. \square

We would like to express the $N_f^{(l)}$ as Frobenius pullbacks of a coherent sheaf. This coherent sheaf is expressed in terms of the test ideals of f .

Let $\lambda \in \mathbb{R}_{\geq 0}$ and let $\tau(f^\lambda) \subset \mathcal{O}_X$ be the test ideal of exponent λ of f . The test ideals form a decreasing sequence of ideals $\subset \mathcal{O}_X$. A *F-jumping exponent* of f is a positive real number $\lambda \in \mathbb{R}_{\geq 0}$, such that, $\forall \epsilon > 0$, $\tau(f^{\lambda-\epsilon}) \neq \tau(f^\lambda)$, see [12, Section 3] for definitions. They satisfy the following finiteness theorem:

Theorem 1. ([3]) *The set of F-jumping exponents of f is*

- (1) *a discrete subset of \mathbb{R}*
- (2) *a subset of \mathbb{Q} .*

For all $l \in \mathbb{N}$, recall that the Frobenius pullback $(F^{l+1})^*M$ of an \mathcal{O}_X -module M is canonically endowed with a structure of left $D_X^{(l)}$ -module. Then one has a canonical isomorphism of left $D_X^{(l)}$ -modules coming from an elementary equality of ideals ([12, Proof of Lemma 6.8.]):

$$D_X^{(l)} f^n \cong (F^{l+1})^* \tau(f^{\frac{n}{p^{l+1}}}).$$

Thus, by Proposition 1, $\forall l \in \mathbb{N}$,

$$N_f^{(l)} \cong (F^{l+1})^* \left(\bigoplus_{0 \leq n < p^{l+1}} \tau(f^{\frac{n}{p^{l+1}}}) / \tau(f^{\frac{n+1}{p^{l+1}}}) \right).$$

For l large enough, this direct sum is concentrated at the F -jumping exponents of f .

Indeed, for $\epsilon > 0$ very small, let $\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda))$ be the direct sum over the F -jumping exponents of f in $(0, 1]$:

$$\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda)) := \bigoplus_{\lambda \in (0,1]} \tau(f^{\lambda-\epsilon}) / \tau(f^\lambda).$$

The sum is finite since by (1) of Theorem 1, there are only finitely many F -jumping exponents of f in $(0, 1]$.

One has,

Proposition 2. *Let $l \in \mathbb{N}$ be large enough so that $\forall n, 0 \leq n < p^{l+1}$, each interval $(\frac{n}{p^{l+1}}, \frac{n+1}{p^{l+1}}]$ contains at most one F -jumping exponent of f . Then there is a canonical isomorphism of left $D_X^{(l)}$ -modules,*

$$N_f^{(l)} \cong (F^{l+1})^* (\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda))).$$

Proof. The proof is clear. \square

We can now prove the

Theorem 2. *Let k be a field of positive characteristic p , X a smooth variety over k and $f : X \rightarrow \mathbb{A}^1$ a nonconstant function on X . Then the left D_X -module N_f is a finitely generated unit F -module.*

Proof. By Proposition 2, the left D_X -module N_f is expressed as an inductive limit of Frobenius pullbacks of a coherent \mathcal{O}_X -module,

$$N_f = \varinjlim_{l \gg 0} N_f^{(l)} \cong \varinjlim_{l \gg 0} (F^{l+1})^*(\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda))).$$

Hence it is a finitely generated unit F -module. \square

2. THE RELATION TO TEST IDEALS

In order to understand the finer properties of the b -function and its roots, we want to study further its relationship to test ideals.

2.1. Preliminaries on p -adic and $\frac{1}{p}$ -adic expansions. We will express the p -adic expansion of the roots of the b -function of f in terms of the $\frac{1}{p}$ -adic expansion of the F -jumping exponents of f . Let us first introduce some definitions.

Definition 4. *The $\frac{1}{p}$ -adic expansion of a positive real number r is its unique base p expansion which does not have infinitely many consecutive zero coefficients, $r = \sum_{-b \leq n} r_n (\frac{1}{p})^n$.*

Let us focus our attention on properties of the $\frac{1}{p}$ -adic expansion of rational numbers in $(0, 1]$. We say that the $\frac{1}{p}$ -adic expansion of a number r is *periodic* if $\exists l \geq 0, \exists d \geq 1$ and $\{b_1, \dots, b_l, a_1, \dots, a_d\} \subset \{0, \dots, p-1\}$ such that

$$r = \frac{b_1}{p} + \dots + \frac{b_l}{p^l} + \frac{a_1}{p^{l+1}} + \dots + \frac{a_d}{p^{l+d}} + \frac{a_1}{p^{l+d+1}} + \dots + \frac{a_d}{p^{l+2d}} + \frac{a_1}{p^{l+2d+1}} + \dots$$

If $l = 0$, this means that the $\frac{1}{p}$ -adic expansion is of the form:

$$r = \frac{a_1}{p} + \dots + \frac{a_d}{p^d} + \frac{a_1}{p^{d+1}} + \dots + \frac{a_d}{p^{2d}} + \frac{a_1}{p^{2d+1}} + \dots$$

It is then called *strictly periodic*. The minima for l and d are called the *length of the preperiod* of r and the *length of the period* of r , respectively.

In the following lemma, we characterize the periodicity properties of $\frac{1}{p}$ -adic expansions.

Lemma 1. (1) *The $\frac{1}{p}$ -adic expansion of a positive real number r is periodic if and only if r is rational.*

(2) *The $\frac{1}{p}$ -adic expansion of a rational number $r \in (0, 1]$ is strictly periodic if and only if $r \in \mathbb{Z}_{(p)}$.*

Proof. The proof is straightforward and left to the reader. \square

To each rational number $r \in (0, 1] \cap \mathbb{Z}_{(p)}$ correspond finitely many conjugated rational numbers $\in \mathbb{Z}_{(p)}$:

Definition 5. Let r be a positive rational number. By Lemma 1, its $\frac{1}{p}$ -adic expansion is periodic. Let $a_1 a_2 \cdots a_d$ be its period. The $\frac{1}{p}$ -conjugates of r are the d positive rational numbers whose $\frac{1}{p}$ -adic expansion is obtained by replacing the period by its cyclic permutations $\{a_1 a_2 \cdots a_d, a_d a_1 a_2 \cdots, \dots, a_2 \cdots a_d a_1\}$.

Let us now point out another property of strictly periodic $\frac{1}{p}$ -adic expansions.

Lemma 2. Let s be a sequence $s : \mathbb{N}_0 \rightarrow \{0, \dots, p-1\}$, and let $\Lambda \subset (0, 1] \cap \mathbb{Q}$ be a finite set. Suppose that $\forall N \gg 0, \exists \lambda \in \Lambda$ such that the $\frac{1}{p}$ -adic expansion of λ starts by

$$\lambda = \frac{s(N)}{p} + \frac{s(N-1)}{p^2} + \cdots + \frac{s(1)}{p^N} + \dots$$

Then $\exists L \geq 0$ such that $\forall n \geq L, \exists \mu \in \Lambda \cap \mathbb{Z}_{(p)}$ such that

$$\mu = \frac{s(n)}{p} + \frac{s(n-1)}{p^2} + \cdots + \frac{s(1)}{p^n} + \dots$$

Proof. Let us, $\forall l \geq 1$, denote by s_l the number $s_l := \frac{s(l)}{p} + \frac{s(l-1)}{p^2} + \cdots + \frac{s(1)}{p^l}$ and let us say that the $\frac{1}{p}$ -adic expansion of a number r starts by s_l , if

$$r = \frac{s(l)}{p} + \frac{s(l-1)}{p^2} + \cdots + \frac{s(1)}{p^l} + \dots$$

By the finiteness of $\Lambda, \exists \lambda \in \Lambda$ such that λ starts by s_l , for arbitrary large l . Also, $\exists L \geq 0$ such that, $\forall l \geq L$, if $\alpha \in \Lambda$ starts with s_l , then it is of this type. Let us show that the $\frac{1}{p}$ -adic expansion of such a λ is strictly periodic.

By Lemma 1.1, the $\frac{1}{p}$ -adic expansion of λ is periodic. Let l be the length of the preperiod of λ . Choose $n \geq l$ and $N \geq n+l$ such that λ starts by s_n and s_N . That is,

$$\begin{aligned} \lambda &= \frac{b_1}{p} + \cdots + \frac{b_l}{p^l} + \frac{a_1}{p^{l+1}} + \cdots + \frac{a_d}{p^{l+d}} + \frac{a_1}{p^{l+d+1}} + \cdots \\ &= \frac{s(n)}{p} + \cdots + \frac{s(n-l+1)}{p^l} + \frac{s(n-l)}{p^{l+1}} + \cdots + \frac{s(1)}{p^n} + \frac{a_*}{p^{n+1}} + \cdots \\ &= \frac{s(N)}{p} + \cdots + \frac{s(N-l+1)}{p^l} + \frac{s(N-l)}{p^{l+1}} + \cdots + \frac{s(n)}{p^{N-n+1}} + \cdots + \frac{s(n-l+1)}{p^{N-n+l}} + \cdots + \frac{s(1)}{p^N} + \frac{a_{*'}}{p^{N+1}} + \cdots \end{aligned}$$

Thus, by the uniqueness of the $\frac{1}{p}$ -adic expansion of λ , the second equality implies that $(b_1, \dots, b_l) = (s(n), \dots, s(n-l+1))$. This combines with the third equality to imply that (b_1, \dots, b_l) is a subsegment of $(a_1, a_2, \dots, a_d, a_1, \dots)$. Thus the $\frac{1}{p}$ -adic expansion of λ is strictly periodic.

Since the $\frac{1}{p}$ -adic expansion of λ is strictly periodic, each of the $\mu \in \Lambda \cap \mathbb{Q}$ starting by s_n for arbitrary large n is a $\frac{1}{p}$ -conjugate of λ . Thus μ has a strictly periodic $\frac{1}{p}$ -adic expansion and hence $\mu \in \Lambda \cap \mathbb{Z}_{(p)}$. This concludes the proof of the lemma. \square

We then relate $\frac{1}{p}$ -adic and p -adic expansions of opposite rational numbers.

Lemma 3. *Let r be a rational number, $r \in (0, 1] \cap \mathbb{Z}_{(p)}$, and let $r = \sum_{1 \leq n} r_n (\frac{1}{p})^n$ be its $\frac{1}{p}$ -adic expansion. By Lemma 1, it is strictly periodic. Let d be the length of its period. The p -adic expansion of $-r$ is*

$$-r = \sum_{1 \leq n} r_{d-(n \bmod d)} p^{n-1}.$$

Proof. The proof is straightforward and left to the reader. \square

2.2. More on the unit F -structure of N_f . We would like to understand better the unit F -structure of N_f and show that the higher Euler operators are compatible with its Frobenius endomorphism. To do so, we study the isomorphism

$$N_f = \varinjlim_{l \geq 0} N_f^{(l)} \cong \varinjlim_{l \geq 0} (F^{l+1})^*(\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda)))$$

from the proof of Theorem 2.

We first refine Proposition 2.

Proposition 3. *Let $l \in \mathbb{N}$ be large enough so that $\forall n, 0 \leq n < p^{l+1}$, each interval $(\frac{n}{p^{l+1}}, \frac{n+1}{p^{l+1}}]$ contains at most one F -jumping exponent of f and let λ be a F -jumping exponent of f in $(0, 1]$. Then the composition of the isomorphisms of Propositions 1 and 2,*

$$\bigoplus_{0 \leq n < p^{l+1}} D_X^{(l)} f^n / D_X^{(l)} f^{n+1} \cong N_f^{(l)} \cong (F^{l+1})^*(\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda)))$$

induces an isomorphism

$$D_X^{(l)} f^m / D_X^{(l)} f^{m+1} \cong (F^{l+1})^* \tau(f^{\lambda-\epsilon}) / \tau(f^\lambda),$$

for the unique m such that $\frac{m}{p^{l+1}}$ is the truncated $\frac{1}{p}$ -adic expansion of λ .

Proof. It is clear that $\lambda \in (\frac{n}{p^{l+1}}, \frac{n+1}{p^{l+1}}]$ if and only if the $\frac{1}{p}$ -adic expansion of λ starts by $\frac{n}{p^{l+1}}$. \square

Let us now study the structure maps of the inductive system

$$N_f \cong \varinjlim_{l \geq 0} (F^{l+1})^*(\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda))).$$

Proposition 4. *Let d be the lcm of the lengths of the periods of the $\frac{1}{p}$ -adic expansions of the F -jumping exponents of f in $(0, 1] \cap \mathbb{Z}_{(p)}$. Then $\exists e > 0$, a multiple of d , such that $\forall l \in \mathbb{N}$ as in Proposition 3 and $\forall \lambda$ F -jumping exponent of f in $(0, 1]$, the e -th iterate of the structure maps:*

$$(F^{l+1})^*(\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda))) \rightarrow (F^{l+e+1})^*(\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda)))$$

induces a morphism:

$$(F^{l+1})^*(\tau(f^{\lambda-\epsilon})/\tau(f^\lambda)) \rightarrow (F^{l+e+1})^*(\tau(f^{\lambda-\epsilon})/\tau(f^\lambda)).$$

Moreover, this morphism vanishes if $\lambda \notin \mathbb{Z}_{(p)}$.

Proof. By Proposition 3 and the description of the structure maps in Proposition 1, $\forall i > 0$, the i -th iterate of the structure maps sends

$$(F^{l+1})^*(\tau(f^{\lambda-\epsilon})/\tau(f^\lambda)) \rightarrow \bigoplus_{0 \leq n < p^{l+i+1} \text{ and } n \equiv m \pmod{p^{l+1}}} D_X^{(l+i)} f^n / D_X^{(l+i)} f^{n+1},$$

for the unique m such that $\frac{m}{p^{l+1}}$ is the truncated $\frac{1}{p}$ -adic expansion of λ .

By Proposition 3 and Lemma 2, the finiteness of the number of F -jumping exponents of f implies that, $\exists N > 0$ independent of l , such that, $\forall \lambda$ a F -jumping exponent of f in $(0, 1]$ but not in $\mathbb{Z}_{(p)}$, the above map for $i \geq N$ vanishes. Indeed, there is no n as above such that $\frac{n}{p^{l+i+1}}$ is the truncation of the $\frac{1}{p}$ -adic expansion of a F -jumping exponent of f .

Suppose that $\lambda \in \mathbb{Z}_{(p)}$. Then it is straightforward to see that for i a positive multiple of d ,

$$\exists! n, 0 \leq n < p^{l+i+1} \text{ and } n \equiv m \pmod{p^{l+1}}$$

such that $\frac{n}{p^{l+i+1}}$ is the truncation of the $\frac{1}{p}$ -adic expansion of a F -jumping exponent of f in $(0, 1] \cap \mathbb{Z}_{(p)}$. Furthermore, that F -jumping exponent is λ .

Thus if $e' \geq N$ is a multiple of d , then $e = e' + d$ fulfills the proposition. \square

We can now prove the

Theorem 3. *There is an integer $e > 0$ such that, $\forall l \in \mathbb{N}$ as in Proposition 3, the natural injection*

$$\mathbf{gr}_{\lambda \in (0,1] \cap \mathbb{Z}_{(p)}}(\tau(f^\lambda)) \rightarrow \mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda))$$

induces an isomorphism of unit F^e -modules:

$$\varinjlim_{j \geq 0} (F^{l+1+ej})^*(\mathbf{gr}_{\lambda \in (0,1] \cap \mathbb{Z}_p}(\tau(f^\lambda))) \xrightarrow{\sim} \varinjlim_{j \geq 0} (F^{l+1+ej})^*(\mathbf{gr}_{\lambda \in (0,1]}(\tau(f^\lambda))) \cong N_f,$$

where the structure maps of the inductive systems are those of Proposition 4. Moreover, the unit F^e -module N_f splits as a direct sum

$$N_f = \bigoplus_{\lambda \in (0,1] \cap \mathbb{Z}_{(p)}} (N_f)_\lambda,$$

where the λ are the F -jumping exponents of f in $(0, 1] \cap \mathbb{Z}_{(p)}$,

$$(N_f)_\lambda := \varinjlim_{j \geq 0} (F^{l+1+ej})^*(\tau(f^{\lambda-\epsilon})/\tau(f^\lambda))$$

and each of the summands is nontrivial,

$$(N_f)_\lambda \neq 0.$$

Proof. The only fact that is not a straightforward consequence of Proposition 4 is the nontriviality of the summands, $(N_f)_\lambda \neq 0$.

However, $(N_f)_\lambda = 0$ implies that the image under the structure map of $f^m \in D_X^{(l)} f^m$ in $D_X^{(l+je)} f^{m'}/D_X^{(l+je)} f^{m'+1}$ is zero, for some $j > 0$, and m and m' as in Proposition 3. But by the description of the structure maps in Proposition 1, as used in the proof of Proposition 4, the image of f^m is a nonzero multiple of $f^{m'}$. Thus its vanishing implies that of the quotient

$$0 = D_X^{(l+je)} f^{m'}/D_X^{(l+je)} f^{m'+1} \cong (F^{l+je+1})^*(\tau(f^{\frac{m'}{p^{l+je+1}}})/\tau(f^{\frac{m'+1}{p^{l+je+1}}}))$$

and hence that of

$$\tau(f^{\frac{m'}{p^{l+je+1}}})/\tau(f^{\frac{m'+1}{p^{l+je+1}}}) = 0.$$

Which, by the definition of m' , is absurd since λ is a F -jumping exponent of f . \square

Remark 2. The decomposition of N_f as a direct sum $N_f = \bigoplus_{\lambda \in (0,1] \cap \mathbb{Z}_{(p)}} (N_f)_\lambda$ is canonical. However, the assignment of a F -jumping exponent to each of the summands is not. Indeed, different choices of l in Theorem 3 exchange the $(N_f)_\lambda$'s, for $\frac{1}{p}$ -conjugated λ 's. This only depends on $l \bmod e$.

The compatibility of the higher Euler operators and the Frobenius endomorphism of N_f is another corollary of Proposition 4. Indeed, we have the

Theorem 4. There is an integer $e > 0$ such that the higher Euler operators act as endomorphisms of the unit F^e -module N_f .

Proof. We take the same $e > 0$ as in Theorem 3. By this very Theorem 3, it is enough to show the result for each of the $(N_f)_\lambda$, say for $l+1$ a positive multiple of the lcm of the lengths of the periods of the $\frac{1}{p}$ -adic expansions of the F -jumping exponents of f in $(0, 1] \cap \mathbb{Z}_{(p)}$. However by Propositions 1 and 3, since λ has a strictly periodic $\frac{1}{p}$ -adic expansion,

$$\lambda = \frac{a_1}{p} + \dots + \frac{a_d}{p^d} + \frac{a_1}{p^{d+1}} + \dots + \frac{a_d}{p^{2d}} + \frac{a_1}{p^{2d+1}} + \dots,$$

$\forall i \in \mathbb{N}$, the action of ν_i on $(N_f)_\lambda$ is by $-a_{d-(i \bmod d)}$. Thus $(N_f)_\lambda$ is a common eigenspace of the higher Euler operators. This proves the theorem. \square

Corollary 1. *Let k be a perfect field of positive characteristic p , X a smooth variety over k and $f : X \rightarrow \mathbb{A}^1$ a nonconstant function on X . Then the b -function of f has finitely many roots.*

Proof. We show that there are only finitely many maximal ideals of $k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$ containing the ideal $b_f \subset k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$. By the proof of Remark 1, the maximal ideals of $k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$ are defined over \mathbb{F}_p . Thus it is enough to show that there are only finitely many maximal ideals of $\mathbb{F}_p[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$ containing $b_f \cap \mathbb{F}_p[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$. The latter is the annihilator of N_f in $\mathbb{F}_p[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$.

By the Riemann-Hilbert correspondence for unit F -modules ([5]), the unit F^e -module N_f corresponds to an object in the constructible derived category of étale \mathbb{F}_p -sheaves on X . Since by Theorem 4, the algebra $\mathbb{F}_p[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]$ acts on N_f by endomorphisms of unit F^e -module, it is transported by the Riemann-Hilbert correspondence to act on an object in the constructible derived category of étale \mathbb{F}_p -sheaves on X . The algebra of global endomorphisms of those being finite dimensional over \mathbb{F}_p , the algebra $\mathbb{F}_p[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]/b_f$ is finite dimensional over \mathbb{F}_p . It has thus finitely many maximal ideals, which proves the corollary. \square

2.3. The roots of the b -function and F -jumping exponents. Here we describe the roots of the b -function of f in terms of its F -jumping exponents.

Theorem 5. *Let k be a field of positive characteristic p , X a smooth variety over k and $f : X \rightarrow \mathbb{A}^1$ a nonconstant function on X . The roots of the b -function of f are the opposites of the F -jumping exponents of f which are in $(0, 1] \cap \mathbb{Z}_{(p)}$.*

Proof. By Theorem 3,

$$b_f := \text{ann}_{k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]}(N_f) = \bigcap_{\lambda \in (0, 1] \cap \mathbb{Z}_{(p)}} \text{ann}_{k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]}(N_f)_\lambda,$$

where the λ are the F -jumping exponents of f in $(0, 1] \cap \mathbb{Z}_{(p)}$.

For $l+1$ in Theorem 3 a positive multiple of the lcm of the lengths of the periods of the $\frac{1}{p}$ -adic expansions of the F -jumping exponents of f in $(0, 1] \cap \mathbb{Z}_{(p)}$, the action of the higher Euler operators on the $(N_f)_\lambda$'s is computed in the proof of Theorem 4. Indeed, if λ has strictly periodic $\frac{1}{p}$ -adic expansion

$$\lambda = \frac{a_1}{p} + \dots + \frac{a_d}{p^d} + \frac{a_1}{p^{d+1}} + \dots + \frac{a_d}{p^{2d}} + \frac{a_1}{p^{2d+1}} + \dots,$$

then, $\forall i \in \mathbb{N}$, the action of ν_i on $(N_f)_\lambda$ is the multiplication by $-a_{d-(i \bmod d)}$. Thus the action of $(\frac{s}{p^i})$ is the multiplication by $a_{d-(i \bmod d)}$ and $\text{ann}_{k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]}(N_f)_\lambda$ is the maximal ideal

$$\text{ann}_{k[(\frac{s}{p^0}), (\frac{s}{p^1}), (\frac{s}{p^2}), \dots]}(N_f)_\lambda = \left(\left(\frac{s}{p^i} \right) = a_{d-(i \bmod d)} \mid i \in \mathbb{N} \right).$$

It corresponds by Remark 1 to the p -adic integer

$$\tilde{\lambda}_l = \sum_{i \geq 0} a_{d-(i \bmod d)} p^i.$$

Hence, by Lemma 3, $\tilde{\lambda}_l$ is the opposite of a $\frac{1}{p}$ -conjugate of λ . Thus by Remark 2, $\tilde{\lambda}_l$ is the opposite of a F -jumping exponent of f in $(0, 1] \cap \mathbb{Z}_{(p)}$, and all their opposites are obtained this way. This concludes the proof of the theorem. \square

Remark 3. *Since there are only finitely many F -jumping exponents of f in $(0, 1]$ by Theorem 1, this reproves Corollary 1.*

The following is a direct consequence of Theorem 5:

Corollary 2. *The roots of the b -function are negative rational numbers, ≥ -1 .*

3. EXAMPLES

We now use Theorem 5 to compute some examples of the set of roots of the b -function.

Example 1. *Let $f = x \in \mathbb{F}_p[x]$. Then the set of roots of the b -function of f is $\{-1\}$. Note that the Bernstein-Sato polynomial of $x \in \mathbb{C}[x]$ is $(s+1)$.*

Example 2. *Let $f = x_1^2 + \dots + x_n^2 \in \mathbb{F}_p[x_1, \dots, x_n]$, where $n \geq 2$ and $p > 2$. Then the only F -jumping exponent of f in $(0, 1]$ is 1 ([12, Example 6.16.]). Thus by Theorem 5, the roots of the b -function of f are $\{-1\}$. Note that the Bernstein-Sato polynomial of $x_1^2 + \dots + x_n^2 \in \mathbb{C}[x_1, \dots, x_n]$ is $(s + \frac{n}{2})(s+1)$ ([9, Example 6.2.]).*

Example 3. *Monomial.*

Let $n \geq 1$. And $\forall j, 1 \leq j \leq n$, let α_j be an integer, $\alpha_j \geq 1$. Then it is well-known that the F -jumping exponents of $f = x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{F}_p[x_1, \dots, x_n]$ in $(0, 1]$ are

$$\bigcup_{1 \leq j \leq n} \left\{ \frac{l}{\alpha_j} \mid 1 \leq l \leq \alpha_j \right\}.$$

Thus by Theorem 5, the set of roots of the b -function of $f = x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{F}_p[x_1, \dots, x_n]$ is

$$\bigcup_{1 \leq j \leq n} \left\{ -\frac{l}{\alpha_j} \mid 1 \leq l \leq \alpha_j \right\} \cap \mathbb{Z}_{(p)}.$$

By [9, Lemma 6.10.], the Bernstein-Sato polynomial of $x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathbb{C}[x_1, \dots, x_n]$ is $\prod_{1 \leq j \leq n} \prod_{1 \leq l \leq \alpha_j} (s + \frac{l}{\alpha_j})$.

Example 4. *The cusp.*

Let $f_p = x^2 + y^3 \in \mathbb{F}_p[x, y]$, with $p > 3$. By [12, Example 6.14.], the F -jumping exponents of f_p in $(0, 1]$ are:

$$\left\{\frac{5}{6}, 1\right\} \text{ if } p \equiv 1 \pmod{3}$$

and

$$\left\{\frac{5}{6} - \frac{1}{6p}, 1\right\} \text{ if } p \equiv 2 \pmod{3}.$$

Hence by Theorem 5, the roots of b_{f_p} are:

$$\left\{-1, -\frac{5}{6}\right\} \text{ if } p \equiv 1 \pmod{3}$$

and

$$\{-1\} \text{ if } p \equiv 2 \pmod{3}.$$

Note that by [9, Example 6.19], the Bernstein-Sato polynomial of $f = x^2 + y^3 \in \mathbb{C}[x, y]$ is

$$b_f = \left(s + \frac{7}{6}\right)(s + 1)\left(s + \frac{5}{6}\right).$$

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HIGHER SCHOOL OF ECONOMICS, MOSCOW
E-mail address: tbitoun@gmail.com